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Prize and incentives in double-elimination tournaments

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HIGHLIGHTS

- I studied a game-theoretical model of double-elimination tournaments.
- Compared to single-elimination tournaments, players have a second chance to compete.
- The standard version produces higher total effort than single-elimination.
- The variant version however may produce lower total effort than single-elimination.
- Granting a second chance to symmetric players may create asymmetrical incentives.

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ABSTRACT

I examine a game-theoretical model of two variants of double-elimination tournaments, and derive the equilibrium behavior of symmetric players and the optimal prize allocation assuming a designer aims to maximize total effort. I compare these theoretical properties to the well-known single-elimination tournament.

1. Introduction

The use of elimination tournaments is prevalent in job promotions, political elections, sports contests, and so forth. Since the seminal contribution of Rosen (1986), models of elimination tournaments have been frequently used to rationalize the phenomenon of highly concentrated rewards only on the top ranks. For instance, CEOs are often paid disproportionally higher salaries than their immediate subordinates. A presidential election is another extreme example of allocating rewards: winner-takes-all. The economics literature has focused on one particular form of elimination tournaments, single-elimination tournaments, in which competitors who lose in one round are eliminated from the rest of the competition. However, in many situations, competitors are often permitted a second chance. For instance, in job promotion competitions, organizations do not always adopt the so-called "up or out" strategy but allow employees to try again next time. Second chance in elimination tournaments can be captured in a stylized model of double-elimination tournaments in which a competitor who loses in one round still has a second chance to become the final winner.

This note presents a game-theoretical model of two variants of double-elimination tournaments in a four-player case. By comparing to single-elimination tournaments, I investigate how does granting a second chance affect the strategic behavior of symmetric players as well as the optimal prize allocation assuming that a tournament designer aims to maximize total effort. The two variants differ slightly in terms of whether everyone or almost everyone has a second chance.

With the exception of a recent experimental study by Deck and Kimbrough (2015) who examined a similar four-player double-elimination tournament in which each round is resolved as an all-pay auction, economists have largely neglected this tournament format. In contrast, Statisticians have been interested in double-elimination tournaments for decades (e.g., Searls, 1963; Schwertman et al., 1991; Edwards, 1996). Using primarily

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simulations with specific assumptions on probability of winning in each round, they asked which tournament format is better able to select the best among all players. Unlike economists, they rarely considered players' strategic incentives.

2. The model

2.1. Single-elimination tournament

Consider $2n$ risk-neutral players. Let e_{si} be the irreversible effort expended by player i when there are at most s rounds ahead. Let P_s be player i's probability of winning in a round against player j that follows the generalized Tullock contest success function:

$$P_s = \frac{e_{si}^r}{e_{si}^r + e_{sj}^r}.$$

Assume each player incurs a linear cost of effort $c(x) = x$. I focus on a restricted class of games where the contest impact parameter $r \in (0, 1]$.[1]

The winner of the final round is awarded the top prize W_1, the second place player receives the prize W_2, losers of the semifinals are both awarded $W_3, \ldots$, and all players who are eliminated in a round where at most s rounds remain are awarded the prize W_{s+1}. Define the prize spread $\Delta W_s = W_s - W_{s+1}$.

Denote V_s the valuation of winning for player i in a round against player j when s rounds are ahead. Then the recursive objective function for i is

$$V_s = \max_{e_{si}}[P_s V_{s-1} + (1 - P_s)W_{s+1} - c(e_{si})].$$

The pure-strategy equilibrium in the round can be derived as

$$e_{si} = e_{sj} = \frac{r}{4}(V_{s-1} - W_{s+1}).$$

Inserting the equilibrium effort back into the objective function gives

$$V_s - W_{s+1} = \beta(V_{s-1} - W_{s+1}),$$

where $\beta = \frac{1}{2} - \frac{r}{4}$ and $V_0 = W_1$. Also note that

$$V_s - W_{s+2} = \beta(V_{s-1} - W_{s+1}) + \Delta W_{s+1}.$$

In a four-player case, using these two recursive equations gives us the equilibrium efforts in the final and semi-final rounds

$$\begin{cases} e_1 = \frac{r}{4}\Delta W_1, \\ e_2 = \frac{r}{4}(\beta \Delta W_1 + \Delta W_2). \end{cases}$$

Therefore, the total effort in the four-player single-elimination tournament is

$$2e_1 + 4e_2 = \left(r - \frac{r^2}{4}\right)\Delta W_1 + r\Delta W_2,$$

subject to the budget constraint $W_1 + W_2 + 2W_3 = V$, or $\Delta W_1 + 2\Delta W_2 + 4\Delta W_3 = V$. Since $r \le 1, 2(r - \frac{r^2}{4}) > r$. Hence, to maximize the total effort, it is optimal to allocate all budget on the top prize. As such, the optimal total effort becomes

$$T_{single} = \left(r - \frac{r^2}{4}\right)V = \frac{r}{2}(1 + 2\beta)V.$$

[1] The generalized Tullock contest success function covers a wide range of interesting cases from partially discriminating contests to varying degrees ($r > 0$) to perfectly discriminating contests ($r = \infty$, all-pay auction), in all of which the player exerting higher effort has a higher probability of winning. By focusing on the restricted class of games, I make sure the existence of pure-strategy equilibrium in each round. Furthermore, the case in which $r = \infty$ has been studied in Deck and Kimbrough (2015).

2.2. Standard double-elimination tournament

Double-elimination tournaments have two variants. Fig. 1 shows a schematic representation of the double-elimination tournament in a four-player case. In the standard version, there is only one round in the Grand Finals, whereas in the variant version, which will be analyzed in the next section, there are at most two rounds in the Grand Finals.

All four players start from the Upper Bracket. Those who lose in Round 1 fall into the Lower Bracket. Then one of the two players in the Lower Bracket must be eliminated by competing against each other. In Round 2, those who remain in the Upper Bracket compete for a seat in the Grand Finals. The dash line in the Lower Bracket denotes the player who falls into the Lower Bracket in Round 2 and she must compete against the survivor in the Lower Bracket for another seat in the Grand Finals. In Round 3 two finalists compete for the first place award. In the standard double-elimination tournament, they complete only once. In the variant version, if the finalist from the Upper Bracket loses, she has a second chance to compete against the same opponent in Round 4.

The winner of the Grand Finals is awarded the top prize W_1, the second place player receives the prize W_2, the third place player who got eliminated in Round 2 is awarded W_3, and the last place player who got eliminated in Round 1 gets W_4. Define the prize spread $\Delta W_s = W_s - W_{s+1}$. Let $e_{q_{ni}}$ and $e_{q_{nj}}$ denote the effort level exerted by players i and j in Round n and Stage q that is either Upper ($q = U$) or Lower ($q = L$) Bracket or Grand Finals ($q = F$).

Denote V_{q_n} the valuation of winning for player i against player j in Round n and Stage q. The game produces a system of recursive objective functions, which can be worked out backwardly.

In Round 3 and in the Grand Final, F_3, player i has the following objective function:

$$V_{F_3} = \max_{e_{F_{3i}}}[P_{F_3} W_1 + (1 - P_{F_3})W_2 - c(e_{F_{3i}})].$$

Solving this maximization produces the pure-strategy equilibrium:

$$e_{F3} = \frac{r}{4}\Delta W_1.$$

Inserting them back into the objective function gives

$$V_{F_3} - W_2 = \beta \Delta W_1, \quad (1)$$

where $\beta = \frac{1}{2} - \frac{r}{4}$.

In Round 2, the matches in the Upper and Lower Brackets need separate considerations. For the players in the Lower Bracket, L_2, their objective function is

$$V_{L_2} = \max_{e_{L_2}}[P_{L_2} V_{F_3} + (1 - P_{L_2})W_3 - c(e_{L_2})],$$

and solution

$$e_{L_2} = \frac{r}{4}(V_{F_3} - W_3); \qquad V_{L_2} - W_3 = \beta(V_{F_3} - W_3). \quad (2)$$

For the players in the Upper Bracket, U_2, their objective function becomes

$$V_{U_2} = \max_{e_{U_2}}[P_{U_2} V_{F_3} + (1 - P_{U_2})V_{L_2} - c(e_{U_2})],$$

and solution

$$e_{U_2} = \frac{r}{4}(V_{F_3} - V_{L_2}); \qquad V_{U_2} - V_{L_2} = \beta(V_{F_3} - V_{L_2}). \quad (3)$$

Finally, in Round 1, for the players in the Lower Brackets, L_1, their objective function is

$$V_{L_1} = \max_{e_{L_1}}[P_{L_1} V_{L_2} + (1 - P_{L_1})W_4 - c(e_{L_1})],$$

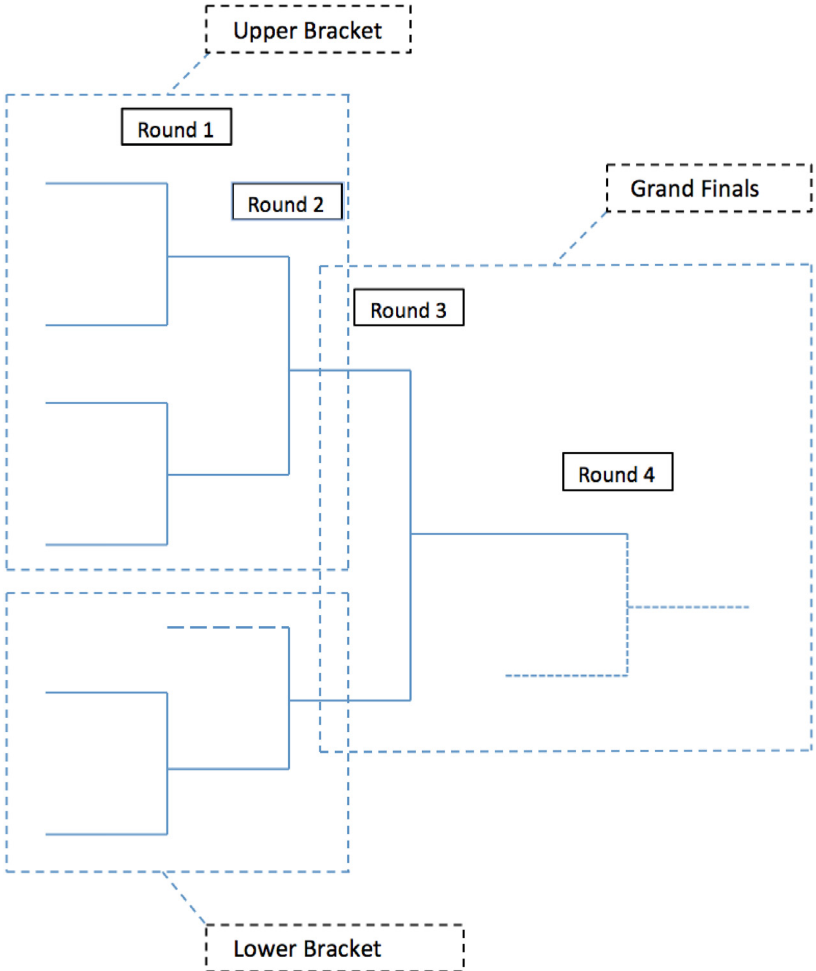

Fig. 1. A representation of the double-elimination tournament in a four-player game.

and solution

$$e_{L_1} = \frac{r}{4}(V_{L_2} - W_4); \qquad V_{L_1} - W_4 = \beta(V_{L_2} - W_4). \tag{4}$$

For the players in the Upper Bracket, U_1, their objective function becomes

$$V_{U_1} = \max_{e_{U_1}}[P_{U_1}V_{U_2} + (1 - P_{U_1})V_{L_1} - c(e_{U_1})],$$

and solution

$$e_{U_1} = \frac{r}{4}(V_{U_2} - V_{L_1}); \qquad V_{U_1} - V_{L_1} = \beta(V_{U_2} - V_{L_1}). \tag{5}$$

Summing up these equilibrium efforts and recursive equations (1)–(5), we can solve all equilibrium efforts as functions of the prize spreads:

$$\begin{cases} e_{F_3} = \frac{r}{4}(\Delta W_1) \\ e_{L_2} = \frac{r}{4}(\beta\Delta W_1 + \Delta W_2) \\ e_{U_2} = \frac{r}{4}(\beta(1-\beta)\Delta W_1 + (1-\beta)\Delta W_2) \\ e_{L_1} = \frac{r}{4}(\beta^2\Delta W_1 + \beta\Delta W_2 + \Delta W_3) \\ e_{U_1} = \frac{r}{4}(2\beta^2(1-\beta)\Delta W_1 + 2\beta(1-\beta)\Delta W_2 + (1-\beta)\Delta W_3). \end{cases}$$

Therefore, the total effort in the four-player standard double-elimination tournament is

$$\begin{aligned} &2(e_{F_3} + e_{L_2} + e_{U_2} + e_{L_1} + 2e_{U_1}) \\ &\quad = \frac{r}{2}\{\Delta W_1(1 + 2\beta + 4\beta^2 - 4\beta^3) + \Delta W_2(2 + 4\beta - 4\beta^2) \\ &\quad\quad + \Delta W_3(3 - 2\beta)\}, \end{aligned}$$

subject to the budget constraint $W_1 + W_2 + W_3 + W_4 = V$, or $\Delta W_1 + 2\Delta W_2 + 3\Delta W_3 + 4\Delta W_4 = V$. Since $r \leq 1$ and $1/4 \leq \beta < 1/2$, $2(1 + 2\beta + 4\beta^2 - 4\beta^3) > 2 + 4\beta - 4\beta^2$ and $3(1 + 2\beta + 4\beta^2 - 4\beta^3) > 3 - 2\beta$. The total-effort-maximizing prize allocation is to place all budget on the top prize. As such, the optimal total effort is

$$T_{Sdouble} = \frac{r}{2}(1 + 2\beta + 4\beta^2 - 4\beta^3)V.$$

2.3. Variant double-elimination tournament

Consider the four-player variant double-elimination tournament. In Round 4 and the second match of the Grand Finals, F_4, each player has the following objective function

$$V_{F_4} = \max_{e_{F_4}}[P_{F_4}W_1 + (1 - P_{F_4})W_2 - c(e_{F_4})].$$

We know that $P_{F_4} = 1/2$ and $e_{F_4} = \frac{r}{4}\Delta W_1$, which in equilibrium lead to

$$V_{F_4} - W_2 = \beta\Delta W_1. \tag{6}$$

In Round 3, consider the two finalists separately; one finalist has climbed from the Upper Bracket and the other from the Lower Bracket. Denote the probability of winning and the valuation of winning the current round for the finalist from the Upper Bracket as $P^U_{F_3}$ and $V^U_{F_3}$, and those for the other finalist from the Lower Bracket as $P^L_{F_3}$ and $V^L_{F_3}$. Thus for the advantaged finalist, the objective function is

$$V^U_{F_3} = \max_{e^U_{F_3}}[P^U_{F_3}W_1 + (1 - P^U_{F_3})V_{F_4} - c(e^U_{F_3})].$$

For the disadvantaged finalist, the objective function becomes

$$V^L_{F_3} = \max_{e^L_{F_3}}[P^L_{F_3}V_{F_4} + (1 - P^L_{F_3})W_2 - c(e^L_{F_3})].$$

Note that $P^U_{F_3} + P^L_{F_3} = 1$, $P^U_{F_3} = \frac{(W_1-V_{F_4})^r}{(W_1-V_{F_4})^r+(V_{F_4}-W_2)^r} > \frac{1}{2}$, and define $\beta^U_{F_3} = P^U_{F_3}(1-rP^U_{F_3})$; $\beta^L_{F_3} = P^L_{F_3}(1-rP^L_{F_3})$, which in equilibrium lead to

$$\begin{aligned} e^U_{F_3} &= rP^U_{F_3}(1 - P^U_{F_3})[W_1 - V_{F_4}]; \\ V^U_{F_3} - V_{F_4} &= \beta^U_{F_3}[W_1 - V_{F_4}], \end{aligned} \tag{7}$$

$$\begin{aligned} e^L_{F_3} &= rP^U_{F_3}(1 - P^U_{F_3})[V_{F_4} - W_2]; \\ V^L_{F_3} - W_2 &= \beta^L_{F_3}[V_{F_4} - W_2]. \end{aligned} \tag{8}$$

The derivations of equilibrium efforts and recursive equations in Rounds 2 and 1 are similar to those in the standard version, and details are omitted here. In Round 2, for the players in the Lower Bracket,

$$e_{L_2} = \frac{r}{4}[V^L_{F_3} - W_3]; \qquad V_{L_2} - W_3 = \beta(V^L_{F_3} - W_3). \tag{9}$$

For the players in the Upper Bracket,

$$e_{U_2} = \frac{r}{4}[V^U_{F_3} - V_{L_2}]; \qquad V_{U_2} - V_{L_2} = \beta(V^U_{F_3} - V_{L_2}). \tag{10}$$

In Round 1, for the players in the Lower Bracket,

$$e_{L_1} = \frac{r}{4}[V_{L_2} - W_4]; \qquad V_{L_1} - W_4 = \beta(V_{L_2} - W_4). \tag{11}$$

For the players in the Upper Bracket,

$$e_{U_1} = \frac{r}{4}[V_{U_2} - V_{L_1}]; \qquad V_{U_1} - V_{L_1} = \beta(V_{U_2} - V_{L_1}). \tag{12}$$

Summing up these equilibrium efforts and recursive equations (6)–(12), we can solve all equilibrium efforts as functions of the prize spreads

$$\begin{cases} e_{F_4} = \frac{r}{4}\Delta W_1, \\ e^U_{F_3} = rP^U_{F_3}(1 - P^U_{F_3})(1 - \beta)\Delta W_1, \\ e^L_{F_3} = rP^U_{F_3}(1 - P^U_{F_3})\beta\Delta W_1, \\ e_{L_2} = \frac{r}{4}[\beta^L_{F_3}\beta\Delta W_1 + \Delta W_2], \\ e_{U_2} = \frac{r}{4}[(\beta^U_{F_3}(1 - \beta) + \beta - \beta^L_{F_3}\beta^2)\Delta W_1 + (1 - \beta)\Delta W_2], \\ e_{L_1} = \frac{r}{4}[\beta^L_{F_3}\beta^2\Delta W_1 + \beta\Delta W_2 + \Delta W_3], \\ e_{U_1} = \frac{r}{4}[(\beta^U_{F_3}\beta(1 - \beta) + \beta^2 + (1 - 2\beta)\beta^2\beta^L_{F_3})\Delta W_1 \\ \quad +2\beta(1 - \beta)\Delta W_2 + (1 - \beta)\Delta W_3], \end{cases}$$

where $P^U_{F_3} = \frac{(1-\beta)^r}{(1-\beta)^r+(\beta)^r}$. Therefore, the total effort in the variant double-elimination tournament is

$$\begin{aligned} T_{Vdouble} &= e^U_{F_3} + e^L_{F_3} + 2(1 - P^U_{F_3})e_{F_4} \\ &\quad + 2(e_{L_2} + e_{U_2} + e_{L_1} + 2e_{U_1}) \\ &= \frac{r}{2}\{\Delta W_1[(2P^U_{F_3} + 1)(1 - P^U_{F_3}) + \beta^L_{F_3}(\beta + 2\beta^2 - 4\beta^3) \\ &\quad + \beta^U_{F_3}(1 + \beta - 2\beta^2) + \beta + 2\beta^2] \\ &\quad + \Delta W_2[2 + 4\beta - 4\beta^2] + \Delta W_3[3 - 2\beta]\}. \end{aligned} \tag{13}$$

Note that the weights on ΔW_2 and ΔW_3 are exactly the same in the two versions of double-elimination tournaments. However, I cannot unambiguously conclude the total-effort-maximizing prize allocation is to place all the budget on the top prize. Specifically, there is a small region of r close to 1 where it is optimal to maximize the prize spread ΔW_2.

In the first match of the Grand Finals, the finalist from the Upper Bracket works harder than the opposing finalist from the Lower Bracket. This implies that even among a homogeneous population a selection effect is at work. Player who fall in the Lower Bracket will have to win more matches to reach the Grand Finals than the finalist who has never lost. Importantly, even if one of them survives through the Lower Bracket and reaches the Grand Finals, she no longer enjoys another chance to lose while her opposing finalist does.

2.4. Comparisons

Table 1
Parallel comparisons of equilibrium efforts in each match.

Match	Effort comparisons
F_3, F_4	$e_{F_3} = \tilde{e}_{F_4}$
L_2	$e_{L_2} > \tilde{e}_{L_2}$
U_2	$e_{U_2} < \tilde{e}_{U_2}$
L_1	$e_{L_1} > \tilde{e}_{L_1}$
U_1	$e_{U_1} < \tilde{e}_{U_1}$

Parallel comparisons of the equilibrium efforts in the standard and the variant double-elimination tournaments reveal that the standard version gives more incentives to players in the Lower Bracket and less incentives to those in the Upper Bracket precisely because of the distorted weights on ΔW_1. Intuitively, given that the finalist from the Upper Bracket will have no second chance, it becomes less attractive to be in that position. The results are summarized in Table 1 (e for the standard, $\tilde{e}$ for the variant).

To compare the total efforts between the two versions, recall that only their weights on ΔW_1 differ. Unfortunately, there is no unambiguous result from this comparison. For example, if $r \to 0$, then $P^U_{F_3} = \frac{1}{2}$, $\beta^U_{F_3} = \beta^L_{F_3} = \beta = \frac{1}{2}$ and $T_{Sdouble} < T_{Vdouble}$. But if $r = 1$, then $P^U_{F_3} = \frac{3}{4}$, $\beta^U_{F_3} = \beta^L_{F_3} = \frac{3}{16} < \beta = \frac{1}{4}$ and $T_{Sdouble} > T_{Vdouble}$.

A conjecture is that there exists a value of r^* in the region of (0, 1] such that if $r > r^*$ then $T_{Sdouble} > T_{Vdouble}$, but if $r < r^*$ then $T_{Sdouble} < T_{Vdouble}$. A numerical simulation supports this conjecture and the threshold r^* lies near 0.8. Thus, when the contest success function is sufficiently discriminating, a tournament designer can induce higher total effort in the standard version than in the variant version in the four-player case, even though the standard version has fewer matches.

Finally, I can unambiguously rank the total effort in single- and double-elimination tournaments when $r = 1$ (i.e. probability of winning is proportional to relative effort): $T_{Sdouble} > T_{single} > T_{Vdouble}$. In fact, $T_{Sdouble} > T_{single}$ holds any value of r.

3. Conclusion

In this note, I examined a model of double-elimination tournaments. In the four-player case, the standard version produces higher total effort level than a single-elimination tournament with the same prize budget. As in the single-elimination tournament, a designer could maximize total effort by allocating all budget to the first place prize. In the variant version, granting a second chance to every player creates asymmetrical motivations for the two finalists to win the first place prize. Furthermore, even within the restricted class of games, it is not always true that the total-effort-maximizing prize allocation is to place all budget on the first place prize. It is also not possible to unambiguously rank the total effort level between the standard and the variant version of double-elimination tournaments without fixing a value for the contest impact parameter; it is possible that the variant version may produce lower total effort level than a single-elimination tournament, even though it has twice as many matches.

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